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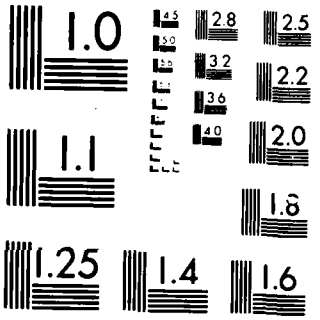

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OPERATIONS RESEARCH CENTER

THE GROUP CONSENSUS PROBLEM

by

Kiduck Chang*

ORC 85-13

December 1985

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ABSTRACT

In a group consensus problem, there is a group with $K \geq 2$ members who are jointly responsible for the aggregation of their opinions. The group may or may not have a predefined real decision problem. French [1983] called the group consensus problem with a predefined real decision problem a *group decision problem* and the group consensus problem without a real decision problem a *text-book problem*.

Suppose a group with K members are interested in forecasting demands for a commodity for a given time period. Production planning for this commodity depends on demands. Each group member may have his own opinion for demands in the form of probability distribution. In this case, the group has a real decision problem in which they should determine the amount of the commodity to be produced. Here the group consensus opinion is a probability distribution for demands obtained from the group members' prior opinions for demands.

On the other hand, a group may simply be required to give their opinions for others to use at some time in the future in as yet undefined circumstances. Here, there is no predefined decision problem. For example, a group of meteorologists are required to give a single forecast for weather without having any real decision problem. This is an example of the text-book problem. Savage [1954] suggested that the whole of statistical theory is directly or indirectly aimed at the solution of a version of the text-book problem.

The objective of this paper is to give a unified approach for these two problems. In this paper all the group members are assumed to be Bayesians. Keywords: 1473

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THE GROUP CONSENSUS PROBLEM

1.1. Introduction

In a group consensus problem, there is a group with $K \geq 2$ members who are jointly responsible for the aggregation of their opinions. The group may or may not have a predefined real decision problem. French [1983] called the group consensus problem with a predefined real decision problem a *group decision problem* and the group consensus problem without a real decision problem a *text–book problem*.

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The objective of this paper is to give a unified approach for these two problems. In this paper all the group members are assumed to be Bayesians.

The Group Decision Problem

Suppose the group is faced with a decision problem with an action space A and a state space Θ. The group members must jointly determine a decision or action from A. The outcome of any action depends on the state of the world $\theta \in \Theta$. Each member provides his beliefs

and preferences by a subjective probability for θ, p_i, and utility function, v_i, defined on $A \times \Theta$, respectively. Thus each member has a preference ordering $\preccurlyeq'$ for actions in A defined by

$$a_1 \preccurlyeq' a_2 \quad \text{iff} \quad E_i\, v_i(a_1, \theta) \leq E_i\, v_i(a_2, \theta), \tag{1.1}$$

where E_i denotes expectation with respect to the probability distribution p_i.

Most approaches (e.g., Bacharach [1975]) to this problem assume the existence of a group preference ordering $\preccurlyeq^G$ on A such that there is a probability distribution p_G and a utility function v_G satisfying

$$a_1 \preccurlyeq^G a_2 \quad \text{iff} \quad E_G\, v_G(a_1, \theta) \leq E_G\, v_G(a_2, \theta),$$

where E_G denotes expectation with respect to p_G. However, Arrow's Impossibility Theorem (Arrow [1951], Kelly [1978]) shows that there is no fair way of forming a group preference ordering from the individual preference orderings alone. One interpretation of Arrow's Impossibility Theorem is that, in general, there is no procedure for combining individual preference orderings into a group preference ordering that does not explicitly address the question of interpersonal comparison of preferences (Keeney and Raiffa [1976]). Hence Arrow's Impossibility Theorem requires that some constraints be given on the possible forms of the individual preference orderings in order to obtain a group preference ordering which is consistent with the seemingly innocuous Assumptions given by Arrow. (For these Assumptions, see Arrow [1951] or Keeney and Raiffa [1976]). The restriction on the form of the individual preference ordering, however, does not give a fair rule for combining the individual orderings except in the case that either the members share the same utilities, or they share the same probabilities (Raiffa [1968], Bacharach [1975]). Here "fair" means that the group consensus opinion should satisfy the Pareto Optimality Principle and the group consensus opinion can not be a single individual's opinion. The Pareto Optimality Principle is satisfied if there exists no alternative decision that some member would find better and none would find worse.

Bacharach [1975] considers the individual preference orderings derived from the expected utilities as given by equation (1.1). But he has arrived at an impossibility theorem, which says

that there is no fair way of combining the individual orderings when the individuals disagree on their utilities. When the group members share the same utilities, he showed that the group consensus probability is given by a linear opinion pool. But he needs a set of assumptions for the form of individual preferences, which are too strong in some cases. One of these assumptions is "column linearity", which says that, for any four actions a_k, a_l, $a_{k'}$, $a_{l'}$ in A, if $E_i v_i(a_k,\theta) - E_i v_i(a_l,\theta) = E_i v_i(a_{k'},\theta) - E_i v_i(a_{l'},\theta)$ for all i, then $a_k \leqslant^G a_l$ implies $a_{k'} \leqslant^G a_{l'}$. Hence we can not consider Bacharach's result as a justification for the linear opinion pool in the case where the individuals share the same utilities.

In a different way from Bacharach, de Finetti claimed that, in a group decision problem, a collective action by several individuals, who agree on their evaluations of utility (by reducing it, for instance, to monetary terms) but not on those of the probabilities, must be optimal for a hypothetical individual whose opinion is convexly comprised among those of the real individuals concerned (de Finetti [1972], p. 196). He suggested this conjecture by an example of a simple hypothesis testing problem (de Finetti [*1954*]). But de Finetti's conjecture does not imply that the group consensus opinion must be a convex combination of the group members' opinions. Moreover, de Finetti's conjecture is not necessarily true when the group is not allowed to take a randomized action as will be seen in Example 1.3.

Let X be the set of all probability mass or density functions for θ. For any $x \in X$, the ith group member can determine an action $a_i^*(x) \in A$ which is optimal against x in i's opinion, that is,

$$E_i v_i[a_i^*(x), \theta] = \max_{a \in A} \sum_{\theta} x(\theta) v_i(a, \theta).$$

So for each $x \in X$, there corresponds $a_i^*(x)$ for $i=1, \ldots, K$. Then we can transform the utility v_i on $A \times \Theta$ to a utility function u_i on $X \times \Theta$, which will be discussed in Section 1.2. In this paper we will work with u_i on $X \times \Theta$ rather than v_i on $A \times \Theta$ to determine the group consensus opinion in a group decision problem. In other words, the group determines its consensus opinion from X based on the individual utilities u_i on $X \times \Theta$ and their prior opinions

for θ under the Pareto Optimality Principle. This means that we transform the real decision problem with action space $\mathbf{A}$ and state space Θ to a decision problem with decision space $\mathbf{X}$ and state space Θ. We will discuss this in more detail in Section 1.2.

The Text-Book Problem

One approach to this problem is to introduce a supra decision maker and let him update his beliefs as in the expert problem (Keeney and Raiffa [1976]). However, his updated opinion is his subjective or personal probability. Hence there is no guarantee that the group members will agree with his opinion. His opinion is only data for the individuals concerned.

An alternative way is to let each group member assume the role of the supra decision maker and then report his updated opinion in turn. However, it does not solve the fundamental problem of combining their opinions, if, after several iterations of this process, the opinions of the group have not yet converged (Genest and Zidek [1984]).

The approach suggested in this paper is to treat the text-book problem as a version of the group decision problem. As mentioned before, the group decision problem can be considered as a decision problem with decision space $\mathbf{X}$ and state space Θ, where each group member has a utility function u_i defined on $\mathbf{X} \times \Theta$. In the text-book problem, there are no actions to be chosen by the group. But we can consider a group consensus opinion as a group decision to be determined by the group. It is assumed that each group member has a utility function u_i defined on $\mathbf{X} \times \Theta$. For each $x \in \mathbf{X}$, $u_i(x, \theta)$ is group member i's utility for the group opinion $x \in \mathbf{X}$ when the state of the world is $\theta \in \Theta$. The utility $u_i(\cdot,\cdot)$ can be interpreted as i's evaluation of probability distributions for θ or i's psychological value for the probability distribution which is chosen as a group consensus opinion. Group members will be more satisfied if the group consensus opinion gives high probability for the actual outcome θ.

We can consider the text-book problem as a version of the group decision problem with decision space $\mathbf{X}$ and state space Θ, and individual utility functions u_i defined on $\mathbf{X} \times \Theta$. The only difference between the group decision problem and the text-book problem is that in a

group decision problem u_i is derived from v_i on $\mathbf{A} \times \Theta$, but in a text-book problem u_i is evaluated directly by the ith group member. Hence the utilities u_i in a group decision problem are proper as will be seen in section 1.2, but the utilities u_i in a text-book problem are not necessarily proper.

1.2. Proper Utility Functions

This section is mainly concerned with the group decision problem. Suppose a group of K members has a real decision problem with action space A and state space Θ. Each group member has a utility function v_i defined on $\mathbf{A} \times \Theta$. We will discuss the transformation of the individual utility functions v_i on $\mathbf{A} \times \Theta$ to the utility functions u_i on $\mathbf{X} \times \Theta$. It will be shown that the transformed utility functions u_i are proper.

Definition

A utility function u_i is *proper* if

$$\sum_{\theta} p_i(\theta) u_i(p_i, \theta) \geq \sum_{\theta} p_i(\theta) u_i(x, \theta) \qquad (1.2)$$

for all $x \in \mathbf{X}$, where p_i is i's true opinion for θ.

u_i is *strictly proper* if the inequality in (1.2) is strict for all $x \neq p_i$ in $\mathbf{X}$.

The above definition says that if an individual has a proper utility function, then he will announce his true opinion as his opinion for θ, that is, he is honest in announcing his opinion.

For each $x \in \mathbf{X}$, there corresponds an action $a_i^*(x) \in \mathbf{A}$ such that

$$\sum_{\theta} x(\theta) v_i[a_i^*(x), \theta] = \max_{a \in \mathbf{A}} \sum_{\theta} x(\theta) v_i(a, \theta).$$

Notice that $a_i^*(x)$ is a Bayes action against $x \in \mathbf{X}$ for the ith group member. For convenience we assume that $a_i^*(x)$ is unique for each $x \in \mathbf{X}$. We will relax this assumption later.

Now define a function u_i on $\mathbf{X} \times \Theta$:

$$u_i(x, \theta) = v_i[a_i^*(x), \theta] \quad \text{for } x \in \mathbf{X} \text{ and } \theta \in \Theta. \tag{1.3}$$

We can interpret $u_i(x,\theta)$ as i's utility when i uses distribution $x \in \mathbf{X}$ and θ occurs. Let $p_i \in \mathbf{X}$ be i's true opinion for θ. Then i's expected utility for $x \in \mathbf{X}$ is

$$\begin{aligned}\bar{u}_i(x) &= \sum_\theta p_i(\theta) u_i(x, \theta) \\ &= \sum_\theta p_i(\theta) v_i[a_i^*(x), \theta].\end{aligned}$$

We can interpret $\bar{u}_i(x)$ as i's expected utility when i takes an action which is optimal against x, while his true opinion is p_i. Hence $\bar{u}_i(x)$ is i's expected utility for x when i's true opinion is p_i. Thus we can use u_i defined by (1.3) as i's utility function on $\mathbf{X} \times \Theta$. From now on, we can assume that each member of the group has a utility function u_i defined on $\mathbf{X} \times \Theta$.

Lemma 1.1.: u_i defined by equation (1.3) is proper.

Proof

Suppose i's true opinion is $p_i \in \mathbf{X}$. Then for any $x \in \mathbf{X}$

$$\begin{aligned}\bar{u}_i(x) &= \sum_\theta p_i(\theta) u_i(x, \theta) \\ &= \sum_\theta p_i(\theta) v_i[a_i^*(x), \theta] \\ &\leq \max_{a \in A} \sum_\theta p_i(\theta) v_i(a, \theta) \\ &= \sum_\theta p_i(\theta) v_i[a_i^*(p_i), \theta] \\ &= \sum_\theta p_i(\theta) u_i(p_i, \theta) \\ &= \bar{u}_i(p_i).\end{aligned}$$

This is true for any $p_i \in \mathbf{X}$. Therefore u_i is proper. □

Note that (i) u_i is not necessarily strictly proper, (ii) if v_i's are equivalent up to linear transformations, then u_i's are also equivalent, and (iii) u_i is strictly proper if and only if $a_i^* : \mathbf{X} \to \mathbf{A}$ is one to one.

Definition

We say that two utility functions u and u' on $\mathbf{X} \times \Theta$ are equivalent if there exist real numbers $a > 0$ and b such that $u'(\cdot,\cdot) = au(\cdot,\cdot) + b$

Actually b could depend on θ and the maximizing decisions would still be the same.

Example 1.1.

Let $\Theta = [0, 1]$ and $\mathbf{A} = \{ a : 0 \leq a \leq 1 \}$, where $a \in \mathbf{A}$. Any distribution for θ is represented by a real number $x \in [0, 1]$, where x is a probability for the event $\{\theta = 1\}$. So $\mathbf{X} = [0, 1]$.

Suppose $v_i(a, \theta) = -(a-\theta)^2$, i.e., a negative of the quadratic loss, and i's true opinion for θ is $p_i \in \mathbf{X}$. For any $x \in \mathbf{X}$,

$$\max_{a \in \mathbf{A}} \{x\, v_i(a,1)+(1-x)v_i(a,0)\} = \max_{a \in \mathbf{A}} \{-x(a-1)^2-(1-x)a^2\} = -x(1-x)$$

and $a_i^*(x) = x$. Hence $u_i(x, \theta) = v_i(x, \theta) = -(x-\theta)^2$ for all $x \in \mathbf{X}$ and $\theta \in \Theta$. We can easily check that u_i is strictly proper. □

Example 1.2.

Let $\Theta = [0, 1]$ and $\mathbf{A} = \{ a : 0 \leq a \leq 1 \}$, where $a \in \mathbf{A}$ is an estimate of θ. Let $\mathbf{X}$ be the set of all probability densities for θ. Suppose $v_i(a, \theta) = -(a-\theta)^2$ and i's true opinion for θ is $p_i \in \mathbf{X}$. Then for any $x \in \mathbf{X}$

$$\begin{aligned} \max_{a \in \mathbf{A}} \int v_i(a, \theta) x(\theta) d\theta &= -\min_{a \in \mathbf{A}} \int (a-\theta)^2 x(\theta) d\theta \\ &= -\min_{a \in \mathbf{A}} \{ a^2 - 2aE_x\theta + E_x\theta^2 \} \\ &= -\min_{a \in \mathbf{A}} \{ (a-E_x\theta)^2 + E_x\theta^2 - E_x^2\theta \}, \end{aligned} \tag{1}$$

where E_x denotes expectation with respect to $x \in \mathbf{X}$.

From (1) we can see that $a_i^*(x) = E_x\theta$ for all $x \in \mathbf{X}$; and hence

$$u_i(x, \theta) = v_i[a_i^*(x), \theta] = v_i(E_x\theta, \theta) = -(\theta - E_x\theta)^2 .$$

This utility function u_i is proper, but not strictly proper, because any two probability densities with the same means have the same utilities. □

De Finetti's conjecture (de Finetti [1972]), which was mentioned in Section 1.1, is true if the transformed utility u_i on $\mathbf{X} \times \Theta$ is strictly proper, which will be proved by Theorem 1.3 in Section 1.4. However, his conjecture is not necessarily true if u_i is not strictly proper.

Example 1.3

Consider a group with two members, say 1 and 2, who are faced with a real decision problem with an action space $\mathbf{A} = \{a_1, a_2, a_3\}$ and a state space $\Theta = [0, 1]$. Suppose each group member has same utility function v on $\mathbf{A} \times \Theta$ defined by Table 1.1.

		state	
		0	1
	a_1	3	0
action	a_2	0	3
	a_3	1	1

Table 1.1. Utilities for action- state pairs, $v(a, \theta)$.

Since any distribution for θ can be represented by a real number $x \in [0, 1]$, we have $\mathbf{X} = [0, 1]$, where $x \in \mathbf{X}$ is a probability for the event $[\theta = 0]$.

Let $\bar{v}(a|x)$ be the expected utility of an action $a \in \mathbf{A}$ with respect to $x \in \mathbf{X}$, that is, $\bar{v}(a|x) = x\, v(a, 0) + (1-x)v(a, 1)$. In Figure 1.1 we plotted $\bar{v}(a_i|x)$ for $i = 1, 2, 3$ as functions of $x \in \mathbf{X}$.

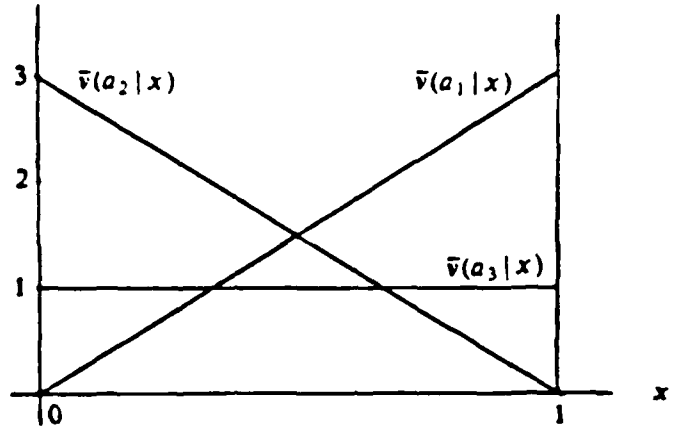

Figure 1.1

Suppose group member 1's [2's] probability for the event $\{\theta = 0\}$ is $\frac{1}{4}$ $[\frac{3}{4}]$. Let $\bar{v}_i(a)$ be i's expected utility for an action a. Then

$$\bar{v}_1(a_1) = \frac{3}{4}\ ,\ \bar{v}_1(a_2) = \frac{9}{4}\ ,\ \bar{v}_1(a_3) = 1$$

and

$$\bar{v}_2(a_1) = \frac{9}{4}\ ,\ \bar{v}_2(a_2) = \frac{3}{4}\ ,\ \bar{v}_2(a_3) = 1.$$

So a_3 is a Pareto optimal action, that is, there is no action $a \in A$ such that

$$\bar{v}_i(a) \geq \bar{v}_i(a_3) \quad \text{for} \quad i = 1,2$$

with strict inequality for at least one i. However, Figure 1.1 indicates that for any $x \in X$, a_3 is not optimal against x. Therefore, de Finetti's conjecture does not hold for this case. □

Definitions

(i) $x \in X$ is a *Pareto optimal* (or *admissible*) decision if there is no $y \in X$ such that $\bar{u}_i(x) \leq \bar{u}_i(y)$ for all $i = 1, \ldots, K$ with strict inequality for at least one i.

(ii) $x \in \mathbf{X}$ is a *Bayes* decision if there exists a $\lambda \in \mathbf{\Lambda}$ such that

$$\sum_{i=1}^{K} \lambda_i \overline{u_i}(x) = \max_{y \in \mathbf{X}} \sum_{i=1}^{K} \lambda_i \overline{u_i}(y),$$

where $\mathbf{\Lambda} = \{ \lambda \in \mathbf{R}^K : \lambda \geq 0 \text{ and } \sum_{i=1}^{K} \lambda_i = 1 \}$.

Let us denote $\mathscr{A}$ as a set of all the Pareto optimal decisions in $\mathbf{X}$ and $\mathscr{B}$ as a set of all the Bayes decisions in $\mathbf{X}$.

In this paper we will prove the following conjectures:

(1) If the group members have equivalent and strictly proper utility functions, then the Pareto optimal decision or the group consensus opinion is a convex combination of their true opinions.

(2) If the group members have equivalent and proper (not necessarily strictly proper) utility functions, then for any decision $x \in \mathbf{X}$ there exists a convex combination of the group members' true opinions, which is at least as good as x to each member of the group.

Conjecture (1) says that $x \in \mathbf{X}$ is Pareto optimal if and only if x is a convex combination of the group members' true opinions. Let $\mathbf{C}$ be the set of all convex combinations of the group members' true opinions, i.e.,

$$\mathbf{C} = \{ x \in \mathbf{X} : x = \sum_{i=1}^{K} \lambda_i p_i \text{ for } \lambda \in \mathbf{\Lambda} \},$$

where p_i is i's true opinion for θ. Conjecture (2) implies that for any $x \in \mathbf{X}$ there is an $x^* \in \mathbf{C}$ such that $\overline{u_i}(x) \leq \overline{u_i}(x^*)$ for all i. This means that a linear opinion pool of the group members' true opinions is a Pareto optimal decision or an optimal group consensus opinion under the Pareto Optimality Principle if the group members have equivalent and proper utility functions. Remember that the utilities of the group members defined by 1.3 in a group decision problem are proper as was shown by Lemma 1.1.

1.3 Single Event Case

Suppose a group with K members are jointly responsible for combining their probabilities for an event A. As assumed before, each group member has a utility function u_i on $\mathbf{X} \times \Theta$, where $\mathbf{X}$ is a space of probability distributions and Θ is a state space. In the single event case, $\Theta = \{ A, \bar{A} \}$ and $\mathbf{X} = [0, 1]$. Each $x \in \mathbf{X}$ denotes a probability for A. Suppose i's true probability for A is p_i and let $\bar{u}_i(x)$ be i's expected utility for $x \in \mathbf{X}$. Then $\bar{u}_i(x) = p_i u_i(x, A) + (1 - p_i) u_i(x, \bar{A})$. Let x_{p_i} be a maximizer of $\bar{u}_i(x)$ over all $x \in \mathbf{X}$, that is,

$$\bar{u}_i(x_{p_i}) = \max_{x \in \mathbf{X}} \bar{u}_i(x) .$$

We assume that i announces his opinion for θ as x_{p_i}. Note that $x_{p_i} = p_i$ if u_i is proper. In this section we will show that, for the single event case, a linear opinion pool of the group members' announced opinions, x_{p_i}, is an optimal group consensus opinion if either (i) the group members' utilities u_i are proper or (ii) the group members' utilities u_i are concave in x. Here u_i, $i = 1, \ldots, K$ need not be equivalent. If u_i are proper for all i, then a linear opinion pool of the group members' *true* opinions is an optimal group consensus opinion. If u_i is *improper*, then $x_{p_i} \neq p_i$. However, a linear opinion pool of x_{p_i}, i's *announced opinion*, for $i = 1, \ldots, K$ is an optimal group consensus opinion if u_i are concave in x for all i.

Lemma 1.2.

If u_i is (strictly) proper, then $\bar{u}_i(x)$ is (strictly) increasing for $0 \leqslant x \leqslant p_i$ and (strictly) decreasing for $p_i \leqslant x \leqslant 1$, where p_i is i's true opinion for A. (Savage [1971], p. 786).

Proof

Let $f_1(x) = u_i(x, A)$ and $f_0(x) = u_i(x, \bar{A})$. Define

$$g(x, p) = p f_1(x) + (1 - p) f_0(x) \quad \text{for } x, p \in \mathbf{X}.$$

Then $g(x, p)$ is i's expected utility for $x \in \mathbf{X}$ when his true probability is p. Let

$0 \leq p_i \leq x_1 < x_2 \leq 1$ and suppose $\bar{u}_i(x_2) > \bar{u}_i(x_1)$. Then, we have

$$p_i[\, f_1(x_2)-f_1(x_1)+f_0(x_1)-f_0(x_2)\,] > f_0(x_1) - f_0(x_2).$$

If u_i is proper, $g(x_1, x_1) \geq g(x_2, x_1)$ and $g(x_2, x_2) \geq g(x_1, x_2)$. However,

$$\begin{aligned} g(x_1,x_1)-g(x_2,x_1) &= x_1 f_1(x_1)+(1-x_1)f_0(x_1)-x_1 f_1(x_2)-(1-x_1)f_0(x_2) \\ &= x_1[f_1(x_1)-f_0(x_1)-f_1(x_2)+f_0(x_2)]+[f_0(x_1)-f_0(x_2)] \\ &< (x_1-p_i)[f_1(x_1)-f_0(x_1)-f_1(x_2)+f_0(x_2)]. \end{aligned}$$

By symmetry, we also have

$$g(x_2, x_2)-g(x_1, x_2) < (x_2- p_i)[f_1(x_2)-f_0(x_2)-f_1(x_1)+f_0(x_1)].$$

Hence, either $g(x_1, x_1) < g(x_2, x_1)$ or $g(x_2, x_2) < g(x_1, x_2)$, which is a contradiction. Therefore, $\bar{u}_i(x_1) \geqslant \bar{u}_i(x_2)$.

In the same way we can show that $\bar{u}_i(x)$ is increasing for $0 \leq x \leq p_i$.

The proof for strictly proper utilities is similar. □

Let us define a set C which consists of convex combinations of the group members' opinions, i.e.,

$$\mathbf{C} = \{\, x \in \mathbf{X} \,:\, x = \sum_{i=1}^{K} \lambda_i p_i \ \text{ for } \ \lambda \in \Lambda \,\}.$$

Theorem 1.1.

(i) Suppose u_i is strictly proper for all i. Then $x \in \mathbf{X}$ is admissible if and only if $x \in \mathbf{C}$.

(ii) Suppose u_i is proper for all i. Then, for any $x \in \mathbf{X}$, there exists $x^* \in \mathbf{C}$ such that $\bar{u}_i(x^*) \geqslant \bar{u}_i(x)$ for all $i = 1, \ldots, K$.

Proof

Suppose u_i is strictly proper for all i and let $0 \leq x < \min_i p_i = p_{i_0}$. Then

$$\begin{aligned} \bar{u}_i(p_{i_0})-\bar{u}_i(x) &= p_i[u_i(p_{i_0}\mathrm{A})-u_i(x,\mathrm{A})] + (1-p_i)[u_i(p_{i_0}\bar{\mathrm{A}})-u_i(x,\bar{\mathrm{A}})] \\ &\geqslant p_{i_0}[u_i(p_{i_0}\mathrm{A})-u_i(x,\mathrm{A})] + (1-p_{i_0})[u_i(p_{i_0}\bar{\mathrm{A}})-u_i(x,\bar{\mathrm{A}})] \end{aligned}$$

$$= p_{i_0} u_i(p_{i_0}, A) + (1-p_{i_0}) u_i(p_{i_0}, \bar{A}) - [p_{i_0} u_i(x, A) + (1-p_{i_0}) u_i(x, \bar{A})]$$
$$> 0.$$

for all $i = 1, \ldots, K$. So any $x < \min_i p_i$ is inadmissible.

Similarly $x > \max_i p_i$ is inadmissible.

Let $\min_i p_i \leq x \leq \max_i p_i$. Then, for any $y < x$, $\bar{u}_i(x) > \bar{u}_i(y)$ for all i with $p_i \geq x$ and, for any $y > x$, $\bar{u}_i(x) > \bar{u}_i(y)$ for all i with $p_i \leq x$. Hence x is admissible. This completes the proof of (i).

The proof of (ii) is similar. □

Example 1.4.

Define Θ, A, and X as in the Example 1.1, that is,

$$\Theta = \{0, 1\}, \quad A = [0, 1], \quad \text{and} \quad X = [0, 1].$$

Remember that $a \in A$ is an estimate of θ and $x \in X$ is a probability for the event $\{\theta = 1\}$. Consider a group of K members with the same utility functions v on A x Θ defined by

$$v(a, \theta) = -(a-\theta)^2 \text{ for } a \in A \text{ and } \theta \in \Theta.$$

Then $u_i(x, \theta) = -(x-\theta)^2$ for all i. By Example 1.1, u_i are strictly proper. Suppose i's true opinion for θ is $p_i \in X$. Then i's expected utility for $x \in X$ is

$$\bar{u}_i(x) = p_i\, u_i(x, 1) + (1-p_i)\, u_i(x, 0)$$
$$= -p_i\,(x-1)^2 - (1-p_i)\, x^2$$

Suppose $K = 2$ and $p_1 = 1/3$ and $p_2 = 2/3$. Then

(i) for any $0 \leq x < p_1$, $\bar{u}_i(x) < \bar{u}_i(p_1)$ for $i = 1, 2$

and

(ii) for any $p_2 < x \leq 1$, $\bar{u}_i(x) < \bar{u}_i(p_2)$ for $i = 1, 2$.

Therefore $x \in X$ is Pareto optimal if and only if $p_1 \leq x \leq p_2$. □

If i's utility function u_i is proper, then his expected utility $\bar{u}_i(x)$ is maximized at $x = p_i$. If, however, u_i is improper, then there exists $x_{p_i} \in \mathbf{X}$ such that

$$\bar{u}_i(x_{p_i}) \geq \bar{u}_i(x) \text{ for all } x \in \mathbf{X} \text{ and } \bar{u}_i(x_{p_i}) > \bar{u}_i(p_i).$$

Here x_{p_i} is a maximizer of i's expected utility. Remember that x_{p_i} is i's announced opinion.

Lemma 1.3.

If u_i is (strictly) concave in x, then $\bar{u}_i(x)$ is (strictly) increasing for $0 \leq x \leq x_{p_i}$ and (strictly) decreasing for $x_{p_i} \leq x \leq 1$, where x_{p_i} is a maximizer of i's expected utility.

Proof

Suppose u_i is concave and let $x_{p_i} \leq x_1 \leq x_2$. Then $x_1 = \lambda x_{p_i} + (1-\lambda)x_2$ for some $0 \leq \lambda \leq 1$ and hence we have

$$\bar{u}_i(x_1) = \bar{u}_i(\lambda x_{p_i} + (1-\lambda)x_2) \geq \lambda \bar{u}_i(x_{p_i}) + (1-\lambda)\bar{u}_i(x_2).$$

Thus

$$\bar{u}_i(x_1) - \bar{u}_i(x_2) \geq \lambda [\, \bar{u}_i(x_{p_i}) - \bar{u}_i(x_2) \,] \geq 0.$$

Similarly, for any $0 \leq x_2 \leq x_1 \leq x_{p_i}$, $\bar{u}_i(x_1) \geq \bar{u}_i(x_2)$.

The proof for the strict concave utility is similar. □

Theorem 1.2.

(i) If u_i is strictly concave in x for all i, then $x \in \mathbf{X}$ is admissible if and only if $x = \sum_{i=1}^{K} \lambda_i x_{p_i}$ for some $\lambda \in \Lambda$.

(ii) If u_i is concave in x for all i, then, for any $x \in \mathbf{X}$, there exists a $\lambda \in \Lambda$ such that $\bar{u}_i(\sum_{i=1}^{K} \lambda_i x_{p_i}) \geq \bar{u}_i(x)$ for all i.

The proof of Theorem 1.2 is similar to the proof of Theorem 1.1.

Theorem 1.2 indicates that the group consensus opinion is a convex combination of their announced opinions if the group members have concave utility functions. Here their announced opinions are not necessarily their true opinions.

We say that an individual with utility function u is *conservative* if

$$p_i \leq x_{p_i} \leq \frac{1}{2} \text{ for } 0 \leq p_i \leq \frac{1}{2}; \text{ and } \frac{1}{2} \leq x_{p_i} \leq p_i \text{ for } \frac{1}{2} \leq p_i \leq 1.$$

We might conjecture that an individual with concave utility function is conservative. But this is not true in general. Suppose $u(x,A) = \sqrt{x}$ and $u(x,A) = \sqrt{1-x}$. Then u is concave in x. However,

$$x_p = \frac{p^2}{p^2 + (1-p)^2} \quad \text{for all } 0 \leq p \leq 1.$$

So, if $p = \frac{3}{4}$, then $x_p = \frac{9}{10}$. Hence an individual with this utility function is not conservative. (See Lindley [1982], p. 7, for further comments on square root utility.)

Suppose u is concave in x and differentiable. Then an individual with utility function u is conservative if and only if

$$x\frac{\partial}{\partial x}u(x, A) + (1-x)\frac{\partial}{\partial x}u(x, \bar{A}) \geq 0 \text{ for } 0 \leq x \leq \frac{1}{2}$$

and

$$x\frac{\partial}{\partial x}u(x, A) + (1-x)\frac{\partial}{\partial x}u(x, \bar{A}) \leq 0 \text{ for } \frac{1}{2} \leq x \leq 1.$$

Note that, if u is proper, then

$$x\frac{\partial}{\partial x}u(x, A) + (1-x)\frac{\partial}{\partial x}u(x, \bar{A}) = 0 \text{ for all } x \in X.$$

However, if an individual is scored by a proper scoring rule for his opinions and his utility for the scores is concave, then he is conservative.

1.4. General Random Variable Case

In this section we will consider the group consensus problem with general state space Θ. As before we define $\mathbf{X}$ as the set of all probability mass or density functions for θ. Also we assume that each group member has a utility function u_i defined on $\mathbf{X} \times \Theta$. Of course in a group decision problem u_i is proper for all i.

In Section 1.4.1, we will show that a linear opinion pool is an optimal group consensus opinion or a Pareto optimal decision in $\mathbf{X}$ if the group members' utility functions u_i are equivalent and proper.

However, if the group members disagree on their utilities and there does not exists a group utility function, we can not have such a strong result as stated above. In Section 1.4.2, we will show that, if the group members' utility functions u_i on $\mathbf{X} \times \Theta$ are concave in x, then any Pareto optimal decision in $\mathbf{X}$ or group consensus opinion is a Bayes decision in $\mathbf{X}$.

In some cases, the group members' utility functions on $\mathbf{X} \times \Theta$ may not be concave. But in Section 1.5 we show that a quadratic approximation of u_i is concave in x. Therefore, if the group members' opinions are close enough and their utility functions u_i are smooth, then we can assume that the group members' utilities are concave in x, at least approximately.

1.4.1. Equivalent and Proper Utility Functions

In Section 1.3, we have considered the group consensus problem in which a group is concerned with a single event. Now we generalize the results of Section 1.3 to a general random variable case. A group with K members are jointly responsible for combining their opinions for an unknown quantity θ or a set of mutually disjoint and exhaustive events. Thus the state space Θ consists of the values which the unknown quantity θ can take or Θ is the set of events under consideration. Each group member's opinion is given by a probability mass or density function for θ. Also each group member has a utility function u_i defined on $\mathbf{X} \times \Theta$. Let $p_i \in \mathbf{X}$ be i's true opinion for θ. Then his expected utility for $x \in \mathbf{X}$ is given by :

$$\bar{u}_i(x) = \begin{cases} \sum_{\theta} p_i(\theta) u_i(x, \theta) & \text{if } \Theta \text{ is discrete.} \\ \int_{\Theta} p_i(\theta) u_i(x, \theta) d\theta & \text{if } \Theta \text{ is continuous.} \end{cases}$$

In this section, we will prove the following conjectures:

(1) If the group members have equivalent and strictly proper utility functions on $X \times \Theta$, then the Pareto optimal decision or the group consensus opinion is a convex combination of their true opinions.

(2) If the group members have equivalent and proper (not necessarily proper) utility functions on $X \times \Theta$, then for any decision $x \in X$ there is a convex combination of the group members' opinions, which is at least as good as x for each group member.

Remember that the group members' utilities defined on $X \times \Theta$ are always proper in the group decision problem. First, we will show that the Pareto optimal decision or the optimal group consensus opinion is a convex combination of the group members' true opinions if the group members have equivalent and strictly proper utility functions.

Theorem 1.3.

If the u_i are equivalent and strictly proper, then $x \in X$ is admissible if and only if $x = \sum_{i=1}^{K} \lambda_i p_i$ for some $\lambda \in \Lambda$.

For the proof of the Theorem 1.3 we need following Definitions and Lemmas.

Definitions

Let S be a subset of R^K.

(i) $s \in S$ is admissible in S if there is no $s' \in S$ such that $s_i \leqslant s_i'$ for all i with strict inequality holding for at least one i, where s_i is ith component of s.

(ii) $s \in S$ is Bayes in S if there exists a $\lambda \in \Lambda$ such that

$$\sum_{i=1}^{K} \lambda_i s_i = \max_{s \in \mathcal{S}} \sum_{i=1}^{K} \lambda_i s_i' .$$

Let us denote $\mathcal{A}_S$ a set of all the admissible points in S and $\mathcal{B}_S$ a set of all the Bayes points in S . Let

$$S = \{ s \in R^K : \text{ there exists an } x \in X \text{ such that } s_i = \bar{u}_i(x) \text{ for all } i \} .$$

Then the set of all Pareto optimal decisions in X and the set of all the Bayes decisions in X defined in Section 1.2 correspond to $\mathcal{A}_S$ and $\mathcal{B}_S$, respectively. That is, $\mathcal{A} = \mathcal{A}_S$ and $\mathcal{B} = \mathcal{B}_S$.

Lemma 1.4.

Let S^* be the convex hull generated by S. Then

(i) $\mathcal{A}_{S^*} \subset \mathcal{B}_{S^*}$.

(ii) If $s^* \in S^*$ is Bayes in S^*, then $s^* = \sum_{j=1}^{r} \xi_j s^{(j)}$, where $r \le K$; $s^{(j)} \in S$ for all $j = 1, \ldots, r$; $\xi_j > 0$ for all j and $\sum_{j=1}^{r} \xi_j = 1$.

Proof: See Blackwell and Girshick [1954].

Under the same conditions of the Theorem 1.3 we have the following Lemmas.

Lemma 1.5: $\mathcal{B}_{S^*} \subset S$.

Proof

Suppose $s^* \in S^*$ is Bayes against $\lambda \in \Lambda$. Then, by Lemma 1.4, $s_i^* = \sum_{j=1}^{r} \xi_j s^{(j)}$, where $r \le K$; $s^{(j)} \in S$ for all $j = 1, \ldots, r$; $\xi_j > 0$ for all j and $\sum_{j=1}^{r} \xi_j = 1$. Moreover, for each j there corresponds $x^{(j)} \in X$ such that $s_i^{(j)} = \bar{u}_i(x^{(j)})$ for all i. Since u_i are equivalent, we can assume that $u_i \equiv u$ for all i, where u is a strictly proper utility function on $X \times \Theta$. Let us define a function $g(x, p)$ by :

$$g(x, p) = \sum \theta p(\theta) u(x, \theta) \quad \text{for } x \text{ and } p \text{ in } \mathbf{X}.$$

Then $g(x, p)$ is linear in p and $g(x, p) < g(p, p)$ for all $x \neq p$. Moreover,

$$\overline{u_i}(x) = \sum_{\theta} p_i(\theta) u(x, \theta) = g(x, p_i).$$

Now we have

$$\begin{aligned}
\sum_{i=1}^{K} \lambda_i s_i^* &= \sum_{i=1}^{K} \lambda_i \sum_{j=1}^{r} \xi_j s_i^{(j)} \\
&= \sum_i \lambda_i \sum_j \xi_j \overline{u_i}(x^{(j)}) \\
&= \sum_i \lambda_i \sum_j \xi_j g(x^{(j)}, p_i) \\
&= \sum_j \xi_j \sum_i \lambda_i g(x^{(j)}, p_i) \\
&= \sum_j \xi_j g(x^{(j)}, \sum_i \lambda_i p_i) \\
&\leq \sum_j \xi_j g(\sum_i \lambda_i p_i, \sum_i \lambda_i p_i) \\
&= g(\sum_i \lambda_i p_i, \sum_i \lambda_i p_i)
\end{aligned}$$

with equality holding if and only if $x^{(j)} = \sum_{i=1}^{K} \lambda_i p_i$ for all j.

Since s^* is Bayes against λ, $x^{(j)} = \sum_{i=1}^{K} \lambda_i p_i$ for all j. So

$$s_i^* = \sum_{j=1}^{r} \xi_j s_i^{(j)} = \sum_{j=1}^{r} \xi_j \overline{u_i}(x^{(j)}) = \sum_{j=1}^{r} \xi_j \overline{u_i}(\sum_{i=1}^{K} \lambda_i p_i) = \overline{u_i}(\sum_{i=1}^{K} \lambda_i p_i)$$

and hence $s^* \in \mathbf{S}$. □

Lemma 1.6: Let $\mathbf{S}_0 = \{s \in \mathbf{S}$: there is a $\lambda \in \mathbf{A}$ s.t. $s_i = \overline{u_i}(\sum_{i=1}^{K} \lambda_i p_i)$ for all $i\}$. Then $\mathcal{C}_{\mathbf{S}} = \mathbf{S}_0$.

Proof

Let $s^0 \in \mathbf{S}_0$. Then there exists an $x \in \mathbf{X}$ such that $x = \sum_{i=1}^{K} \lambda_i p_i$ for some $\lambda \in \mathbf{A}$ and $s_i^0 = \overline{u_i}(x)$ for all i. We have

$$\sum_{i=1}^{K} \lambda_i s_i^0 = \sum_{i=1}^{K} \lambda_i \overline{u_i}(x)$$

$$= \sum_i \lambda_i g(x, p_i)$$

$$= g(x, \sum_i \lambda_i p_i)$$

$$= \max_{y \in X} g(y, \sum_i \lambda_i p_i),$$

because $g(y, \sum_i \lambda_i p_i)$ is maximized at $y = \sum_i \lambda_i p_i$. Thus

$$\sum_{i=1}^{K} \lambda_i s_i^0 = \max_{y \in X} g(y, \sum_i \lambda_i p_i)$$

$$= \max_{y \in X} \sum_i \lambda_i g(y, p_i)$$

$$= \max_{y \in X} \sum_i \lambda_i \sum_\theta p_i(\theta) u(y, \theta)$$

$$= \max_{s \in S} \sum_i \lambda_i s_i$$

$$= \max_{s \in S^*} \sum_i \lambda_i s_i.$$

The last equality follows from Lemma 1.5. Therefore $s^* \in \mathcal{S}_{S^*}$.

Conversely, suppose $s^* \in \mathcal{S}_{S^*}$, then $s^* \in S$ by Lemma 1.5. Hence there is an $x^* \in X$ such that $s_i^* = \overline{u_i}(x^*)$ for all i. However,

$$\sum_{i=1}^{K} \lambda_i s_i^* = \sum_{i=1}^{K} \lambda_i \overline{u_i}(x^*)$$

$$= \sum_i \lambda_i g(x^*, p_i)$$

$$= g(x^*, \sum_i \lambda_i p_i)$$

$$\leq g(\sum_i \lambda_i p_i, \sum_i \lambda_i p_i)$$

with equality holding if and only if $x^* = \sum_i \lambda_i p_i$. Therefore $s^* \in S_0$. □

Lemma 1.7: $S_0 \subset \mathcal{S}_{S^*}$.

Proof

If $s^0 \in S_0$, then there is an $x^0 \in X$ such that $x^0 = \sum_i \lambda_i p_i$ for some $\lambda \in \Delta$ with $s_i^0 = \overline{u_i}(x^0)$ for all i. Suppose s is inadmissible in S^*, then there is an $s^* = \sum_{j=1}^{\infty} \xi_j s^{(j)}$ for some ξ with $\xi \geq 0$ and $\sum_{j=1}^{\infty} \xi_j = 1$ such that (i) $s^{(j)} \in S$ for all j and (ii) $s_i \leq s_i^*$ for all i with holding at least

one strict inequality. Since $s^{(j)} \in \mathbf{S}$ for all j, for each $s^{(j)}$ there corresponds an $x^{(j)} \in \mathbf{X}$ such that $s_i^{(j)} = \overline{u_i}(x^{(j)})$ for all i. Hence

$$\begin{aligned}
\sum_{i=1}^{K} \lambda_i s_i^* &= \sum_{i=1}^{K} \lambda_i \sum_{j=1}^{\infty} \xi_j s^{(j)} \\
&= \sum_i \lambda_i \sum_j \xi_j \overline{u_i}(x^{(j)}) \\
&= \sum_i \lambda_i \sum_j \xi_j g(x^{(j)}, p_i) \\
&= \sum_j \xi_j g(x^{(j)}, \sum_i \lambda_i p_i) \\
&\le \sum_j \xi_j g(\sum_i \lambda_i p_i, \sum_i \lambda_i p_i) \\
&= g(\sum_i \lambda_i p_i, \sum_i \lambda_i p_i) \\
&= \sum_i \lambda_i g(\sum_i \lambda_i p_i, p_i) \\
&= \sum_i \lambda_i g(x^0, p_i) \\
&= \sum_i \lambda_i s_i^0.
\end{aligned}$$

Therefore, $\sum_i \lambda_i s_i^* \le \sum_i \lambda_i s_i^0$ with holding equality if and only if $x^{(j)} = x^0$ for all j. This implies that $s^* = s^0$, which contradicts the assumption that s^0 is dominated by s^*. Hence s^0 is admissible in $\mathbf{S}^*$. □

Proof of Theorem 1.3:

By the previous Lemmas we have $\mathcal{A}_{\mathbf{S}^*} = \mathcal{C}_{\mathbf{S}^*} = \mathbf{S}_0$. Hence $\mathcal{A}_{\mathbf{S}} = \mathcal{A}_{\mathbf{S}^*}$ and $\mathcal{C}_{\mathbf{S}} = \mathcal{C}_{\mathbf{S}^*}$, because $\mathcal{A}_{\mathbf{S}^*} \subset \mathbf{S}$ and $\mathcal{C}_{\mathbf{S}^*} \subset \mathbf{S}$. Thus $\mathcal{A}_{\mathbf{S}} = \mathbf{S}_0$, that is, $s \in \mathbf{S}$ is admissible in $\mathbf{S}$ if and only if $s \in \mathbf{S}_0$. Equivalently, $x \in \mathbf{X}$ is admissible if and only if $x = \sum_i \lambda_i p_i$ for some $\lambda \in \Lambda$. This completes the proof of the theorem. □

We can easily see that de Finetti's conjecture (de Finetti [1972]), which was mentioned in Section 1.3 is true if the group members' utilities u_i are equivalent and strictly proper.

Example 1.5.

Consider a text-book problem in which a group with K members are required to give a probability density for a continuous random variable θ. Let p_i be i's probability density for θ. Suppose each group member i has same utility function u_i on $\mathbf{X} \times \Theta$ for $i = 1, \ldots, K$, where $\mathbf{X}$ is the set of all probability densities for θ. If $u_i(x, \theta) = \log x(\theta)$, then

$$\begin{aligned}\overline{u_i}(x) &= \int u_i(x, \theta) p_i(\theta) d\theta \\ &= \int p_i(\theta) \log x(\theta) d\theta .\end{aligned}$$

Now we have

$$\begin{aligned}\max_{x \in \mathbf{X}} \overline{u_i}(x) &= \max_{x \in \mathbf{X}} \int p_i(\theta) \log x(\theta) d\theta \\ &= \int p_i(\theta) \log p_i(\theta) d\theta ,\end{aligned}$$

and hence u_i is strictly proper.

Since u_i is strictly concave, we can show that $\mathscr{A} = \mathscr{B}$, that is, $x \in \mathbf{X}$ is an admissible decision if and only if x is a Bayes decision. (See Theorem 1.6.) For any $\lambda \in \Lambda$,

$$\begin{aligned}\sum_{i=1}^{K} \lambda_i \overline{u_i}(x) &= \sum_{i=1}^{K} \lambda_i \int p_i(\theta) \log x(\theta) d\theta \\ &= \int [\sum_i \lambda_i p_i(\theta)] \log x(\theta) d\theta \\ &\le \int [\sum_i \lambda_i p_i(\theta)] \log [\sum_i \lambda_i p_i(\theta)] d\theta\end{aligned}$$

with equality holding if and only if $x = \sum_i \lambda_i p_i$.

Therefore $x \in \mathbf{X}$ is an admissible decision if and only if $x \in \mathbf{C}$. □

Now is the time to prove the conjecture that for any decision $x \in \mathbf{X}$ there is a convex combination of the group members' true opinions, which is at least as good as x for each

member of the group. Let

$$C = \{ x \in X : x = \sum_{i=1}^{K} \lambda_i p_i \text{ for some } \lambda \in \Lambda \}.$$

Theorem 1.4.

Suppose u_i are equivalent and proper. Then for any $x \in X$, there exists an $x^* \in C$ such that $\overline{u_i}(x^*) \geq \overline{u_i}(x)$ for all i.

Proof

Without loss of generality we can assume that $u_i \equiv u$ for all i, where u is strictly proper. Let $\epsilon > 0$. We can find a strictly proper utility function s on $X \times \Theta$ such that

$$|s(x, \theta)| < \epsilon/2 \quad \text{for all } x \in X \text{ and } \theta \in \Theta.$$

This can be done by letting

$$s(x, \theta) = \frac{1}{M}[-2x(\theta) + \int_{\theta} x^2(\theta) d\theta]$$

for large $M > 0$. Then s is strictly proper and bounded.

Now define a utility function u^* by

$$u^*(x, \theta) = u(x, \theta) + s(x, \theta)$$

for all x and θ.

We can see that the utility function u^* is strictly proper. Now let

$$\mathcal{A}^* = \{ x \in X : x \text{ is admissible in } X \text{ for } u^* \}$$

Then $\mathcal{A}^*$ is complete, because u^* is bounded from above and the maximizers of $\sum_{i=1}^{K} \lambda_i \overline{u_i^*}(x)$ are always in X for all $\lambda \in \Lambda$ (see Berger [1980]). Therefore, for each $x \in X$, there exists $x^* \in C$ such that $\overline{u_i^*}(x^*) \geq \overline{u_i^*}(x)$ for all i.

From the definition of u^* we have

$$\overline{u_i}(x) + \sum_{\theta} p_i(\theta) s(x, \theta) \leq \overline{u_i}(x^*) + \sum_{\theta} p_i(\theta) s(x^*, \theta)$$

and hence

$$\begin{aligned}\overline{u_i}(x) &\le \overline{u_i}(x^*) + \sum_\theta p_i(\theta)[s(x^*,\theta) - s(x,\theta)] \\ &\le \overline{u_i}(x^*) + \sum_\theta p_i(\theta)\,|\,s(x^*,\theta) - s(x,\theta)\,| \\ &< \epsilon + \overline{u_i}(x^*)\end{aligned}$$

for all $i = 1, \ldots, K$.

Since ϵ is arbitrary, we have the Theorem. □

Example 1.6.

Consider a group decision problem as in the Example 1.2, where $\Theta = [0,1]$, $A = [0,1]$, and X is the set of all probability densities for θ. Suppose there are K group members with same utility functions v_i on $A \times \Theta$ such that $v_i(a,\theta) = -(a-\theta)^2$ for all a, θ, and i. Then the induced utility function u_i on $X \times \Theta$ is $u_i(x,\theta) = -(\theta - E_x\theta)^2$, which is proper but not strictly proper. Here E_x denotes expectation with respect to x. Let p_i be i's opinion for $i = 1, \ldots, K$. Then i's expected utility for $x \in X$ is

$$\begin{aligned}\overline{u_i}(x) &= -E_i(\theta - E_x\theta)^2 \\ &= -E_i\theta^2 + 2E_i\theta\,E_x\theta - E_x^2\theta\,,\end{aligned}$$

where E_i denotes expectation with respect to p_i.

Note that $\overline{u_i}(x) = \overline{u_i}(y)$ if $E_x\theta = E_y\theta$.

Now we want to show that Theorem 1.4 is true for this example. Since u_i is strictly concave in x, $\mathscr{A} = \mathscr{B}$. Let $x \in X \setminus C$, where C is the set of all convex combinations of $p_1, \ldots, p_K$. Without loss of generality we can assume that x is an admissible decision. Then there is a $\lambda \in A$ such that

$$\sum_i \lambda_i \overline{u_i}(x) = \max_{y \in X} \sum_i \lambda_i \overline{u_i}(y)\,.$$

Let $x^* = \sum_i \lambda_i p_i$. Then $\sum_i \lambda_i \overline{u_i}(x) = \sum_i \lambda_i \overline{u_i}(s^*)$, because u_i is proper. Now

$$\sum_i \lambda_i \overline{u_i}(x) = \sum_i \lambda_i (-E_i\theta^2 + 2E_i\theta\,E_x\theta - E_x^2\theta)$$

$$= -\sum_i \lambda_i E_i\theta^2 + 2E_x\theta \sum_i \lambda_i E_i\theta - E_x^2\theta$$

$$= -E_{x^*}\theta^2 + 2E_x\theta\, E_{x^*}\theta - E_x^2\theta$$

and

$$\sum_i \lambda_i \bar{u}_i(x^*) = \sum_i \lambda_i (-E_i\theta^2 + 2E_i\theta\, E_{x^*}\theta - E_{x^*}^2\theta)$$

$$= -\sum_i \lambda_i E_i\theta^2 + 2E_{x^*}\theta \sum_i \lambda_i E_i\theta - E_{x^*}^2\theta$$

$$= -E_{x^*}\theta^2 + 2E_{x^*}^2\theta - E_{x^*}^2\theta$$

$$= -E_{x^*}\theta^2 + E_{x^*}^2\theta.$$

Hence $E_x\theta = E_{x^*}\theta$. So $\bar{u}_i(x) = \bar{u}_i(x^*)$ for all i. □

As a corollary to Theorem 1.4, if there exists a group utility function u_G, which is proper by Lemma 1.1, in a group decision problem, then the linear opinion pool is optimal for the group. Here the group utility function u_G need not be a functional form of individual utility functions u_i, $i = 1, \ldots, K$. Hence Arrow's Impossibility Theorem is not relevant here.

1.4.2. Concave Utility Functions

In Section 1.4.1, we have shown that a linear opinion pool of the group members' opinions is an optimal group consensus opinion or a Pareto optimal decision in X if the group members' utility functions on X x Θ are equivalent and proper. However, if the group members disagree on their utility functions u_i for $i = 1, \ldots, K$ and there does not exist a group utility function, we can not have such a strong result as Theorem 1.4. In the rest of this section we will show that, if the group members' utility functions u_i on X x Θ are concave in x, then any admissible (or Pareto optimal) decision in X is a Bayes decision in X. We will consider the quadratic approximation of the utility functions u_i in the next section and show that the quadratic approximation of u_i is concave in x.

Theorem 1.5.

If u_i, $i = 1, \ldots, K$, are concave in x, then any admissible decision in $\mathbf{X}$ is a Bayes decision in $\mathbf{X}$.

Proof

Let $s^* \in \mathbf{S}^*$ be admissible in $\mathbf{S}^*$. Then, by Lemma 1.4, there is a $\lambda \in \mathbf{A}$ such that s^* is Bayes against λ and s^* is represented by

$$s_i^* = \sum_{j=1}^{r} \xi_j \overline{u}_i(x^{(j)})$$

for some $x^{(1)}, \ldots, x^{(r)}$ and $r \le K$. Since u_i are concave, we have

$$s_i^* \le \overline{u}_i\Big(\sum_{j=1}^{r} \xi_j x^{(j)}\Big) = \overline{u}_i(x^0),$$

where $x^0 = \sum_{j=1}^{r} \xi_j x^{(j)}$. However, $\sum_{i=1}^{K} \lambda_i s_i^* \ge \sum_{i=1}^{K} \lambda_i \overline{u}_i(x^0)$, because s^* is Bayes against λ. Therefore $s_i^* = \overline{u}_i(x^0)$ for all i and hence $s^* \in \mathbf{S}$. This implies that $\mathcal{A}_{\mathbf{S}^*} \subset \mathcal{A}_{\mathbf{S}}$.

Let $s \in \mathcal{A}_{\mathbf{S}}$ and suppose s is inadmissible in $\mathbf{S}^*$. Then there exists $s^* \in \mathbf{S}^*$ such that $s_i \le s_i^*$ for all i with holding at least one strict inequality. Since $s^* \in \mathbf{S}^*$, there is a sequence $x^{(1)}, x^{(2)}, \cdots$ in $\mathbf{X}$ such that $s_i^* = \sum_{j=1}^{\infty} \xi_j \overline{u}_i(x^{(j)})$ for all i. Then

$$s_i^* \le \overline{u}_i\Big(\sum_{j=1}^{\infty} \xi_j x^{(j)}\Big) = \overline{u}_i(x^*),$$

where $x^* = \sum_j \xi_j x^{(j)}$. This means that s is dominated by $(\overline{u}_i(x^*), i = 1, \ldots, K) \in \mathbf{S}$, which is a contradiction. Therefore $\mathcal{A}_{\mathbf{S}^*} = \mathcal{A}_{\mathbf{S}} \subset \mathcal{B}_{\mathbf{S}^*} \cap \mathbf{S}$. It is easy to show that $\mathcal{B}_{\mathbf{S}^*} \cap \mathbf{S} = \mathcal{B}_{\mathbf{S}}$. Hence $\mathcal{A}_{\mathbf{S}} \subset \mathcal{B}_{\mathbf{S}}$. □

Theorem 1.6.

If u_i, $i = 1, \ldots, K$, are strictly concave in x, then $\mathscr{A}' = \mathscr{E}$.

Proof

The proof of the Theorem depends on following Lemmas.

Under the same condition as Theorem 1.6, we have Lemma 1.8 and Lemma 1.9.

Lemma 1.8.

If x and y in X are Bayes against $\lambda \in \Delta$, then $\bar{u}_i(x) = \bar{u}_i(y)$ for all i with $\lambda_i > 0$ if and only if $x = y$.

Proof

Suppose x and y are Bayes against λ and $\bar{u}_i(x) = \bar{u}_i(y)$ for all i with $\lambda_i > 0$. Since u_i are strictly concave, if $x \neq y$, then for any $0 < \alpha < 1$,

$$\bar{u}_i[\alpha x + (1-\alpha)y] > \alpha \bar{u}_i(x) + (1-\alpha)\bar{u}_i(y) = \bar{u}_i(x)$$

for all i with $\lambda_i > 0$. Hence

$$\sum_{i=1}^{K} \lambda_i \bar{u}_i[\alpha x + (1-\alpha)y] > \sum_{i=1}^{K} \lambda_i \bar{u}_i(x),$$

which is a contradiction. So $x = y$.

The converse is trivial. □

Lemma 1.9: If $x \in X$ is Bayes, then x is admissible.

Proof

Suppose x is Bayes against $\lambda \in \Delta$. If $\lambda_i > 0$ for all i, then x is admissible. We can assume that $\lambda_i > 0$ for $i = 1, \ldots, l$ and $\lambda_i = 0$ for $i = l+1, \ldots, K$ for some $1 \leq l < K$.

Suppose x is inadmissible, then there exists $y \in X$ such that

$$\bar{u}_i(y) = \bar{u}_i(x) \text{ for } i = 1, \ldots, l; \text{ and } \bar{u}_i(y) \geq \bar{u}_i(x) \text{ for } i = l+1, \ldots, K$$

with at least one strict inequality holding. By Lemma 1.8, $x = y$, which contradicts the assumption that x is dominated by y. Therefore x is admissible.

This completes the proof of Theorem 1.6. □

By the above Theorems the determination of the group consensus opinion is equivalent to the determination of the Bayes point in $\mathbf{X}$ or the determination of $\lambda \in \mathbf{\Lambda}$, which can be interpreted as the weights given to the group members, if the group members have concave utility functions.

Now we consider the relaxation of the assumption that $a_i^*(x)$, which was defined in Section 1.2, is unique for all i and x. In a group decision problem with action space $\mathbf{A}$, state space $\mathbf{\Theta}$, and individual utility functions v_i on $\mathbf{A} \times \mathbf{\Theta}$, we defined an action $a_i^*(x)$ by

$$\sum_{\theta} x(\theta) v_i[a_i^*(x), \theta] = \max_{a \in \mathbf{A}} \sum_{\theta} x(\theta) v_i(a, \theta)$$

for each $x \in \mathbf{X}$, where $\mathbf{X}$ is the set of all probability mass or density functions for θ. We assumed that $a_i^*(x)$ is unique for each x and defined a utility function u_i on $\mathbf{X} \times \mathbf{\Theta}$ by

$$u_i(x, \theta) = v_i[a_i^*(x), \theta] \text{ for } x \in \mathbf{X} \text{ and } \theta \in \mathbf{\Theta}.$$

Now suppose that $a_i^*(x)$ is not necessarily unique for $x \in \mathbf{X}$. Let δ_i be i's decision rule such that i takes an action $\delta_i(x)$ among the actions which are optimal against x. Then we can define

$$u_i(x, \theta) = v_i[\delta_i(x), \theta] \text{ for } x \in \mathbf{X} \text{ and } \theta \in \mathbf{\Theta}.$$

It is easy to show that the utility function u_i defined as above is proper for any such decision rule δ_i. Furthermore, for any $\delta = (\delta_1, \ldots, \delta_K)$, where δ_i is i's decision rule, the results developed thus far are true. Therefore, if each group member chooses a decision rule δ_i, then we have the same results as the results obtained under the assumption of uniqueness of $a_i^*(x)$.

1.5. Quadratic Approximation

In Section 1.4, we have shown that (i) if the group members have equivalent and proper utility functions, then a Pareto optimal decision or a group consensus opinion is a linear opinion pool, and (ii) if the group members' utilities are concave in x, then a Pareto optimal decision is a Bayes decision. In some cases, however, the concavity assumption is too strong to be satisfied for all group members. The objective of this section is to show that under certain conditions each group member's utility function is nearly concave.

Suppose the state space Θ is finite and suppose that u is proper and differentiable three times. Now consider an individual with utility function u and true opinion $p \in \mathbf{X}$. If Θ consists of n elements then p is an n x 1 vector. His expected utility for $x \in \mathbf{X}$ is

$$\bar{u}(x) = \sum_{\theta} p(\theta) u(x, \theta).$$

The Taylor expansion of $\bar{u}(x)$ is :

$$\bar{u}(x) = \bar{u}(p) + (x-p)'\nabla\bar{u}(p) + \frac{1}{2}(x-p)'H(p)(x-p) + R(x),$$

where $H(p)$ is a Hessian matrix of $\bar{u}(x)$ at $x = p$ and $R(x)$ is the remainder term. Let

$$v(x) = \bar{u}(p) + (x-p)'\nabla\bar{u}(p) + \frac{1}{2}(x-p)'H(p)(x-p).$$

Theorem 1.7.

Suppose the third partial derivatives of $\bar{u}(x)$ are bounded. Then $v(x)$ is concave in x if p is a (relative) interior point of $\mathbf{X}$.

Proof

Since u is proper, $\bar{u}(x)$ is maximized at $x = p$ and hence it should satisfy the Kuhn-Tucker condition :

$$\left[\frac{\partial\bar{u}(x)}{\partial x_j}\right]_{x=p} = \mu$$

for all $j=1, \ldots, n$ for some real number μ. Since $\bar{u}(x) \leqslant \bar{u}(p)$ for all $x \in \mathbf{X}$, we have

$$(x-p)'\nabla\bar{u}(p) + \frac{1}{2}(x-p)'H(p)(x-p) + R(x) \leqslant 0.$$

We know that $\nabla\bar{u}(p) = \mu\mathbf{1}$, where $\mathbf{1}=(1, \ldots, 1)'$. So $(x-p)'\nabla\bar{u}(p) = 0$. Therefore

$$(x-p)'H(p)(x-p) \leqslant -2R(x) \text{ for all } x \in \mathbf{X}. \tag{1}$$

For any $x, y \in \mathbf{X}$ and $0 < \lambda < 1$,

$$v(\lambda x + (1-\lambda)y) \geqslant \lambda v(x) + (1-\lambda)v(x) \text{ iff } (x-y)'H(p)(x-y) \leqslant 0.$$

So we should prove that $(x-y)'H(p)(x-y) \leqslant 0$ for all $x, y \in \mathbf{X}$.

Let $z = p + \frac{1}{M}(x-y)$, where $\max_j \frac{1}{p_j} < M < \infty$. Then $z_j \geqslant 0$ for all j and $\sum_{j=1}^{n} z_j = 1$. So $z \in \mathbf{X}$. By (1)

$$(z-p)'H(p)(z-p) \leqslant -2R(z).$$

So

$$\begin{aligned}(x-y)'H(p)(x-y) &= M^2(z-p)'H(p)(z-p)\\ &\leqslant -2M^2R(z).\end{aligned}$$

Now the remainder term $R(z)$ is

$$R(z) = \frac{1}{3!}\sum_i\sum_j\sum_k (z_i-p_i)(z_j-p_j)(z_k-p_k)\frac{\partial^3}{\partial z_i\partial z_j\partial z_k}\bar{u}[p+\theta(z-p)]$$

$$= \frac{1}{3!}\frac{1}{M^3}\sum_i\sum_j\sum_k (x_i-y_i)(x_j-y_j)(x_k-y_k)\frac{\partial^3}{\partial z_i\partial z_j\partial z_k}\bar{u}[p+\theta(z-p)]$$

for some $0 < \theta < 1$. So

$$M^2R(z) = \frac{1}{6M}\sum_i\sum_j\sum_k (x_i-y_i)(x_j-y_j)(x_k-y_k)\frac{\partial^3}{\partial z_i\partial z_j\partial z_k}\bar{u}[p+\theta(z-p)]$$

and hence $M^2R(z) \to 0$ as $M \to \infty$. Therefore

$$(x-y)'H(p)(x-y) \leqslant 0 \text{ for all } x, y \in \mathbf{X}. \quad \square$$

In Theorem 1.7, we assume that the third partial derivatives of $\bar{u}(x)$ are bounded But, if p is a local maximum point of $v(x)$, then we can show that $v(x)$ is concave without the assumption of Theorem 1.7.

Corollary 1.

If p is a (relative) interior point of **X** and p is a local maximum point of $v(x)$, then $v(x)$ is concave in x.

Proof

For any $x \in \mathbf{X}$ sufficiently close p, $v(x) \leqslant v(p)$. So

$$(x-p)'\nabla\bar{u}(p) + \frac{1}{2}(x-p)'H(p)(x-p) \leqslant 0.$$

Since $\nabla\bar{u}(p) = \mu\mathbf{1}$, $(x-p)'\nabla\bar{u}(p) = 0$ and hence $(x-p)'H(p)(x-p) \leqslant 0$ for all x sufficiently close to p.

For given x, y in **X** let $z = p + \frac{1}{M}(x-y)$, where $\max_j \frac{1}{p_j} < M < \infty$. Then $z \in \mathbf{X}$ and $(z-p)'H(p)(z-p) \leqslant 0$ for sufficiently large M . So

$$(x-y)'H(p)(x-y) = M^2(z-p)'H(p)(z-p) \leqslant 0$$

for all $x, y \in \mathbf{X}$. □

Thus far we have considered only the proper utility functions. However, we can obtain the same result under some smoothness assumptions for the improper utility functions. Suppose $\bar{u}(x)$ is maximized at $y_0 \neq p$. A Taylor expansion of $\bar{u}(x)$ around y_0 is given by :

$$\begin{aligned}\bar{u}(x) &= \bar{u}(y_0) + (x-y_0)'\nabla\bar{u}(y_0) + \frac{1}{2}(x-y_0)'H(y_0)(x-y_0) + R(x) \\ &= v(x) + R(x),\end{aligned}$$

where

$$v(x) = \bar{u}(y_0) + (x-y_0)'\nabla\bar{u}(y_0) + \frac{1}{2}(x-y_0)'H(y_0)(x-y_0).$$

Corollary 2.

Suppose y_0 is a (relative) interior point of $\mathbf{X}$. Then $v(x)$ is concave in x if (i) $\bar{u}(x)$ satisfies the condition of Theorem 1.7 or (ii) y_0 is a local maximum point of $v(x)$.

Note that (i) if u is quadratic and y_0 lies in interior of $\mathbf{X}$, then u is concave and (ii) the Hessian matrix $H(y_0)$ need not be negative (semi)definite in order that $v(x)$ be concave.

Example 1.7.

Let

$$\bar{u}(x) = -x_1 - x_2 + x_1^2 + 3x_1x_2 + x_2^2.$$

Then $\bar{u}(x)$ is maximized at $x = (\frac{1}{2}, \frac{1}{2})$ and the Hessian matrix at $x = (\frac{1}{2}, \frac{1}{2})$ is

$$H(\tfrac{1}{2}, \tfrac{1}{2}) = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}.$$

Notice that $H(\frac{1}{2}, \frac{1}{2})$ is not negative semi-definite. However,

$$(x-y)'H(\tfrac{1}{2}, \tfrac{1}{2})(x-y) = -2(x_1 - y_1)^2 \leq 0$$

for all $x, y \in \mathbf{X}$. So $\bar{u}(x)$ is concave in $\mathbf{X}$. □

If the group members' opinions are close enough and their utility functions are smooth, we can approximate each member's expected utility $\bar{u}_i(x)$ by a quadratic form :

$$v_i(x) = \bar{u}_i(p_i) + (x-p_i)'\nabla\bar{u}_i(p_i) + \frac{1}{2}(x-p_i)'H_i(p_i)(x-p_i), \tag{1.4}$$

where p_i is i's true opinion. Here we assume that u is proper. But we can replace p_i in equation (1.4) by a maximizer of the expected utility $\bar{u}_i$ if u is not proper. The above Theorems say that $v_i(x)$ is concave. Hence any Pareto optimal decision is a Bayes decision.

Since $(x-p_i)'\nabla\bar{u}_i(p_i) = 0$ for all i and $\bar{u}_i(p_i)$ are constants, we can approximate $\bar{u}_i(x)$ by

$$v_i(x) = (x-p_i)'H_i(p_i)(x-p_i).$$

For $\lambda \in \Lambda$, let

$$V_\lambda(x) = \sum_{i=1}^{K} \lambda_i v_i(x)$$
$$= \sum_{i=1}^{K} \lambda_i (x-p_i)'H_i(p_i)(x-p_i).$$

Suppose $x^* \in X$ maximizes $V_\lambda(x)$. Then there is a real number μ satisfying

$$\nabla V_\lambda(x^*) = \mu 1.$$

This is a Kuhn-Tucker necessary condition for the optimality. Since $V_\lambda(x)$ is concave , it is also sufficient condition for the optimality. Now we have

$$\sum_{i=1}^{K} \lambda_i H_i(p_i)(x^*-p_i) = \mu 1$$

and hence

$$x^* = [\sum_i \lambda_i H_i(p_i)]^{-1} [\sum_i \lambda_i H_i(p_i)p_i - \mu 1]$$
$$= \sum_i A^i(p_i-\mu 1),$$

where $A^i = \lambda_i [\sum_i \lambda_i H_i(p_i)]^{-1} H_i(p_i)$.

Suppose $A^i = [a^i_{kl}, \ k, l=1, \ldots, n]$ for $i=1, \ldots, K$. Then

$$x^*_j = \sum_{i=1}^{K}\sum_{l=1}^{n} a^i_{jl}(p_{il}-\mu)$$
$$= \sum_i \sum_l a^i_{jl}p_{il} - \mu \sum_i \sum_l a^i_{jl}$$
$$= \sum_i \sum_l a^i_{jl}p_{il} - \mu ,$$

because $\sum_i A^i = I$ and hence $\sum_i a^i_{jl} = 1$ and $\sum_i a^i_{jl} = 0$ for $l \neq j$.

Since $\sum_{j=1}^{n} x^*_j = 1$, we have

$$\mu = \frac{1}{n}\Big(\sum_{j}\sum_{i}\sum_{l} a_{jl}^{i} p_{il} - 1 \Big).$$

Thus

$$x_j^* = \sum_{i=1}^{K}\sum_{l=1}^{n} a_{jl}^{i} p_{il} - \frac{1}{n}\Big(\sum_{i=1}^{n}\sum_{l=1}^{K}\sum_{l=1}^{n} a_{il}^{i} p_{il} - 1 \Big)$$

$$= \sum_{i=1}^{K}\sum_{l=1}^{n}\Big(a_{jl}^{i} - \frac{1}{n}\sum_{i=1}^{n} a_{il}^{i} + \frac{1}{nK} \Big) p_{il}$$

$$= \sum_{i=1}^{K}\sum_{l=1}^{n} \alpha_{jl}^{i} p_{il}, \qquad (1.5)$$

where $\alpha_{jl}^{i} = a_{jl}^{i} - \frac{1}{n}\sum_{i=1}^{n} a_{il}^{i} + \frac{1}{nK}$ and $\sum_{i=1}^{K}\sum_{l=1}^{n} \alpha_{jl}^{i} = 1$ for all $j = 1, \ldots, n$.

We can partition the equation (1.5) by

$$x_j^* = \sum_{i=1}^{K} \alpha_{ij}^{i} p_{ij} + \sum_{i=1}^{K}\sum_{l \neq j} \alpha_{il}^{i} p_{il},$$

where $\sum_{i=1}^{K} \alpha_{ij}^{i} = 1$ and $\sum_{i=1}^{K}\sum_{l \neq j} \alpha_{il}^{i} = 0$. If α_{il}^{i} for $l \neq j$ are small compared with α_{ij}^{i}, then

$$x_j^* \approx \sum_{i=1}^{K} \beta_{ij} p_{ij}, \qquad (1.6)$$

where $\beta_{ij} = \alpha_{ij}^{i}$ and $\sum_{i=1}^{K} \beta_{ij} = 1$ for all j. We call (1.6) a generalized linear opinion pool. (β_{ij} may be negative.)

1.6. Determination of the Pareto Optimal Decision

We have shown that (i) a Pareto optimal decision or a group consensus opinion is a linear opinion pool if the group members have equivalent and proper utility functions u_i, $i=1, \ldots, K$ defined on $X \times \Theta$, and (ii) a Pareto optimal decision is a Bayes decision if the group members' utilities are concave. Unfortunately, the results of the previous sections do not provide a normative rule for choosing the Pareto optimal decision for the group. Now we want to discuss

the determination of the Pareto optimal decision, i.e., the group consensus opinion. However, as Barlow, Mensing, and Smiriga [1984] pointed out, the determination of the Pareto optimal decision is problem dependent. That is, there is no normative rule which can be reasonably applied to all problems. Here we want to suggest two methods for determining the Pareto optimal decision which, I think, are reasonable.

Method 1

Let $L_i(x) = \bar{u}_i(x_{p_i}) - \bar{u}_i(x)$ for $x \in \mathbf{X}$, where x_{p_i} is a maximizer of i's expected utility, $\bar{u}_i(x) = \sum_{\theta} p_i(\theta) u_i(x, \theta)$. Then $L_i(x)$ is i's expected loss-in-utility when the group consensus opinion is x, while his true opinion is p_i (or his announced opinion is x_{p_i}). Let

$$l_i(x) = \frac{L_i(x)}{\bar{u}_i(x_{p_i})} = 1 - \frac{\bar{u}_i(x)}{\bar{u}_i(x_{p_i})}.$$

Then $l_i(x)$ is i's percentage expected loss-in-utility when the group consensus opinion is $x \in \mathbf{X}$.

Now suppose that the group agree to assign weight w_i for the ith member of the group, where $w_i \geq 0$ and $\sum_{i=1}^{K} w_i = 1$. In most cases $w_i = 1/K$ for all i. Then the group may wish to minimize

$$\sum_{i=1}^{K} w_i l_i(x) = \sum_{i=1}^{K} w_i \left[1 - \frac{\bar{u}_i(x)}{\bar{u}_i(x_{p_i})}\right]. \tag{1.7}$$

If $x^* \in \mathbf{X}$ minimizes (1.7), then x^* is Bayes against $\lambda = (\lambda_1, \ldots, \lambda_K)$, where $\lambda_i = \frac{1}{\alpha} \frac{w_i}{\bar{u}_i(x_{p_i})}$ and α is a constant which makes $\sum_{i=1}^{K} \lambda_i = 1$. If u_i are equivalent and proper, then $x^* = \sum_{i=1}^{K} \lambda_i p_i$.

This method normalizes the group members' utilities by

$$v_i(x, \theta) = \frac{u_i(x, \theta)}{\bar{u}_i(x_{p_i})}$$

where ν_i is i's normalized utility function. In this case

$$\max_{x \in X} \nu_i(x) = 1 \quad \text{for all } i.$$

Method 2

In this method we assume that the group members' utilities are proper, that is, each group member's announced opinion is his true opinion. We introduce a supra decision maker N who has the responsibility for determining the Pareto optimal decision or the group consensus opinion. We assume that N does not have his own opinion about θ and he want to minimize his judgement about the group members' expertise or state of knowledge about θ. Barlow, Mensing, and Smiriga [1984] called this assumption as a minimal judgement assumption. Under this assumption, if N were take only one group member's advice, say ith member, then he would simply adopt i's opinion p_i as his decision. However, N needs to make some judgements about the group members because N could not take only one individual's opinion in a group consensus problem. So N's problem is to make a group consensus opinion while minimizing his judgements about the group members.

If N only takes one group member's advice, say i, then N would adopt i's utility function u_i and compute

$$\max_{x \in X} \sum_{\theta} p_i(\theta) u_i(x, \theta). \tag{1.8}$$

Of course N's decision which maximizes (1.8) is p_i. So, if N only takes i's opinion, then we can consider u_i as N's utility function. Now let us define N's utility function. Let $u_N(x,\theta; i)$ be N's utility for the consequence (x,θ) when N only takes i's advice. Then

$$u_N(x, \theta; i) = A_N(i) u_i(x, \theta) + B_N(i) ,$$

where $A_N(i) > 0$ and $B_N(i)$ are coefficients assessed by N to standardize i's utility function. Now N's expected utility (with respect to θ) is

$$u_N(x; i) = A_N(i)\sum_\theta p_i(\theta)u_i(x, \theta) + B_N(i)$$
$$= A_N(i)\bar{u}_i(x) + B_N(i). \qquad (1.9)$$

Let w_i, $i=1, \ldots, K$ be the weights of the group members given by N. Then N's expected utility for $x \in X$ is

$$\bar{u}_N(x) = \sum_{i=1}^{K} w_i A_N(i)\bar{u}_i(x) + \sum_{i=1}^{K} w_i B_N(i)$$
$$= \sum_i w_i A_N(i)\bar{u}_i(x) + B_N. \qquad (1.10)$$

N may wish to maximize (1.10). For this N should assess w_i and $A_N(i)$ for $i = 1, \ldots, K$.

Now consider a special case that the group members' utility functions are proper and local. A utility function u is local if $u(x, \theta) = u(x(\theta), \theta)$ for all $\theta \in \Theta$, that is, the utility for the consequence (x, θ) depends only upon the probability density of the true state and not upon the density of the states which could have obtained but did not. Bernardo [1979] proved that, for a continuous case, a proper and local utility function u must be of the form :

$$u(x, \theta) = A \log x(\theta) + B(\theta),$$

where A is an arbitrary constant and $B(\cdot)$ is an arbitrary function of θ. For the discrete case, see Mathai and Rathie [1975].

If the group members' utility functions are proper and local , then N's utility is given by

$$u_N(x ; i) = A_N(i)\sum_\theta p_i(\theta) \log x(\theta) + B_N(i). \qquad (1.11)$$

The utility $u_N(x; i)$ given by (1.11) is equivalent to

$$u_N(x ; i) = -A_N(i)\sum_\theta p_i(\theta) \log [p_i(\theta)/x(\theta)] + B_N'(i), \qquad (1.12)$$

where $B_N'(i) = B_N(i) + A_N(i)\sum_\theta p_i(\theta)\log p_i(\theta)$. Let

$$I(x:y) = \sum_\theta x(\theta) \log [y(\theta)/x(\theta)] \quad \text{for } x, y \text{ in } X.$$

$I(x:y)$ is called the (directed) divergence of x from y and is a measure of discrepancy (Mathai and Rathie [1975]). Kerridge [1961] interpreted $I(x:y)$ as a measure of the error

made by the observer in estimating a probability density or mass as $y \in X$, which is in fact $x \in X$. So $u_N(x; i)$ in equation (1.12) can be interpreted as an error made by N by taking $x \in X$ while the true distribution is p_i in N's opinion. Thus N's expected utility

$$u_N = \sum_{i=1}^{K} w_i u_N(X; i)$$
$$= -\sum_i A_N(i) \sum_\theta p_i(\theta) \log [p_i(\theta)/x(\theta)] + \sum_i w_i B_N(i),$$

can be thought as an expected error made by N when he takes $x \in X$ as his opinion. Here w_i can be interpreted either as a probability that i's opinion p_i is a true probability density/mass in N's opinion, or as a probability that N takes i's opinion p_i if N were allowed to take only one group member's opinion.

Now return to the general case. Suppose N judges that $w_i = \frac{1}{K}$ for $i = 1, \ldots, K$. This means that N has no preferences among the group members' opinions prior to learning their opinions. It also seems reasonable that N would be indifferent between the consequences (p_i, i) for $i = 1, \ldots, K$, where (x, i) denotes a consequence that N takes i's advice alone but forecasts θ using x. So N would normalize his utility by letting

$$u_N(p_i, i) = 1 \quad \text{for all } i = 1, \ldots, K. \tag{1.13}$$

Suppose also that N judges

$$\min_{1 \le i \le K} u_N(p_s, i) = u_N(p_{s_i}, i) = 0 \quad \text{for all } i = 1, \ldots, K. \tag{1.14}$$

This means that N is also indifferent between the consequences (p_{s_i}, i) for $i = 1, \ldots, K$.

For the alternative interpretation of the equations (1.13) and (1.14), suppose that each group member is asked to evaluate the group members' opinions. Then ith group member may evaluate the opinions by the expected utilities $\bar{u}_i(p_s)$ for $s = 1, \ldots, K$. We have

$$\max_{1 \le i \le K} \bar{u}_i(p_s) = \bar{u}_i(p_i) \quad \text{and} \quad \min_{1 \le i \le K} \bar{u}_i(p_s) = \bar{u}_i(p_{s_i}).$$

Thus the equations (1.13) and (1.14) imply that N standardizes the group members' evaluation for the other group members' opinions by letting

$$\max_s \bar{v}_i(p_s) = 1 \text{ and } \min_s \bar{v}_i(p_s) = 0 \text{ for } i = 1, \ldots, K,$$

where v_i is i's standardized utility function. Let

$$M_i = \max_s \bar{u}_i(p_s) \text{ and } m_i = \min_s \bar{u}_i(p_s).$$

Then from the equations (1.13) and (1.14), we have

$$A_N(i)M_i + B_N(i) = 1$$

and

$$A_N(i)m_i + B_N(i) = 0$$

for $i = 1, \ldots, K$. Hence

$$A_N(i) = 1/(M_i - m_i) \text{ and } B_N(i) = -m_i/(M_i - m_i)$$

for $i = 1, \ldots, K$. N's expected utility for $x \in X$ is

$$u_N(x) = \sum_{i=1}^{K} \frac{1}{K}\left[\frac{1}{M_i - m_i}\bar{u}_i(x) - \frac{m_i}{M_i - m_i}\right]. \tag{1.15}$$

N would take an opinion which maximizes (1.15). If $x^* \in X$ maximizes (1.15), then x^* is a Bayes decision against $\lambda = (\lambda_1, \ldots, \lambda_K)$ with $\lambda_i = 1/(M_i - m_i)$.

If u_i are proper and equivalent, then

$$x^* = \sum_i \lambda_i p_i = \frac{1}{\alpha}\sum_i \frac{p_i}{M_i - m_i},$$

where α is a constant such that $\sum_i \lambda_i = 1$.

Note that, if u_i are proper and local, then

$$M_i - m_i = \max_s I(p_i; p_s) \quad \text{for } i = 1, \ldots, K,$$

where $I(p_i; p_s)$ is a measure of the error made by using p_s when p_i is i's true distribution. $I(p_i; p_s)$ is also interpreted as a measure of the distance from p_i to p_s as assessed by i. This

measure of distance is asymmetric, which seems reasonable. With this in mind we can interpret $(M_i - m_i)$ as a maximum distance from p_i to other group members' opinions as evaluated by i. N gives a weight λ_i to the ith group member proportional to the inverse of the maximum distance assessed by i. Thus i receives small weight if the maximum distance from his opinion to the other members' opinions is large.

1.7. Summary

We have considered a group consensus problem in which the group members are jointly responsible for combining their opinions for an unknown quantity $\theta \in \Theta$. If the group has a predefined real decision problem, we call it a group decision problem. If the group members are simply required to give their opinions fot θ without having any real decision problem, we call it a text-book problem. In this thesis we treat the text-book problem as a version of group decision problem. We assumed that each group member i has a utility function u_i defined on $\mathbf{X} \times \Theta$, where $\mathbf{X}$ is the set of all probality mass or density functions for θ.

In a group decision problem with action space $\mathbf{A}$, state space Θ, and individual utility functions v_i defined on $\mathbf{A} \times \Theta$, i's utility function u_i on $\mathbf{X} \times \Theta$ is derived from v_i on $\mathbf{A} \times \Theta$ and it was shown that u_i is proper.

In a text-book problem, there is no action space $\mathbf{A}$ and hence no individual utility functions v_i on $\mathbf{A} \times \Theta$. But we assume that each group member i has a utility function u_i on $\mathbf{X} \times \Theta$, which is assessed by i.

We summarize the main results according to the state space: single event case and general random variable case.

Single Event Case

(1) If the group members' utility functions u_i on $\mathbf{X} \times \Theta$ are strictly proper for all i, then a Pareto optimal decision in $\mathbf{X}$ must be a convex combination of the group members' true

opinions, i.e., a linear opinion pool of their true opinions.

(2) If u_i are proper for all i, then a linear opinion pool of the group members' true opinions is a Pareto optimal decision in $\mathbf{X}$. Here a Pareto optimal decision may not be a convex combination of the group members' true opinions. However, for any decision $x \in \mathbf{X}$, there exists a convex combination of the group members' opinions which is at least as good as x for each member of the group, that is, there exists $x^* \in \mathbf{X}$ such that x^* is a convex combination of the group members' true opinions and $\bar{u}_i(x^*) \geq \bar{u}_i(x)$ for all i.

(3) If u_i are strictly concave (not necessarily proper) in x for all i, then a Pareto optimal decision in $\mathbf{X}$ must be a linear opinion pool of the group members' announced opinions. Here we assume that each group member i announces the opinion which maximizes his expected utility $\bar{u}_i(x)$ over all $x \in \mathbf{X}$. If u_i is proper, i's announced opinion is his true opinion.

(4) If u_i are concave in x (not necessarily proper) for all i, then a linear opinion pool of the group members' announced opinins is a Pareto optimal decision in $\mathbf{X}$. Moreover, for any decision x in $\mathbf{X}$, there is a convex combination of the group members' announced opinions which is at least as good as x for each member of the group.

General Random Variable Case

(1) If the group members have the equivalent and strictly proper utility functions u_i on $\mathbf{X} \times \Theta$, then any Pareto optimal decision in $\mathbf{X}$ must be a linear opinion pool of the group member' true opinions.

(2) If the group members have the equivalent and proper (not necessarily strictly proper) utility functions u_i, then linear opinion pool of the group members' true opinions is Pareto optimal decision in $\mathbf{X}$. Moreover, for any decision x in $\mathbf{X}$, there is a convex combination of the group members' true opinions which is at least as good as x for each member of the group.

(3) If u_i are concave in x for all i, then Pareto optimal decision in **X** is Bayes decision in **X**.

(4) A quadratic approximation of u_i is concave in x. If the group members' opinions are close enough and their utility functions u_i are smooth, we can assume that u_i is concave in x for all i, at least approximately. Hence Pareto optimal decision in **X** is Bayes decision in **X** in most reasonable cases. We also discuss the form of Pareto optimal decisions for concave utility cases.

Finally, in Section 1.6, I suggested some methods of determining Pareto optimal decision. However, I do not claim that the methods suggested in this paper are normative rules which can be applied to all problems. The determination of Pareto optimal decision is problem dependent.

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